**Claude-Paul BRUTER**

ARPAM & Département de Mathématiques, Université Paris 12, 94010 Créteil, France

# FINE MATHEMATICAL ART THROUGH THE ARPAM PROJECT

The small French city of Maubeuge, located close to the Belgian border, offers the beautiful sight of starred geometric fortifications created by Vauban in the sixteenth century. Within a large domain including some of these fortifications, the City of Maubeuge has expected for a while to set up a wide cultural complex devoted to Geometry, to its internal presence into Nature, into men's activities, among which of course mathematics and art. The idea of this complex they called "the City of Geometries" came first from a gift to Maubeuge of 130 paintings by the international MADI group who says to find his inspiration in some geometry and abstractness. A museum should be erected to show these paintings. This first project has been coupled to an other cultural project giving birth to a quite nice equilibrated project, sealing once more the blending of art and science, more specifically of art and mathematics.

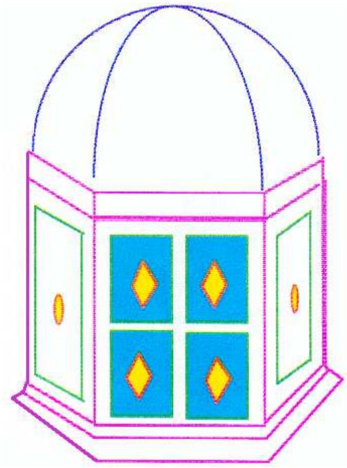
The second project is known as the ARPAM project – Arpam means "association for the realization of the mathematical park". It is a cultural project based on some epistemological considerations in mathematics and in philosophy of sciences, whose scope was first to reconcile the society with mathematics. In this project, mathematical objects of large size, called follies in architectural terms, are used as unusual sculptures, equally as tiny museums for visualization of mathematical concepts, mathematical facts, beautiful mathematical shapes and objects. New shaped trees, gardens, parterres, flower-beds, the diversity of their colours, intend also contribute to illustrate classical geometry, and to offer the restful sight of a picture of a new kind.

As a result, through the exteriorisation of Beauty inside the Mathematics, through a guided tour along well-chosen attractive objects, people of any age may benefit of a deep introduction to mathematics, its rich history, its different significant meanings.

Conversely, through the exteriorisation of Mathematics behind Beauty, people can have the opportunity to admire beautiful luminous objects, to enlarge their vision of the field of the modern artistic creations.

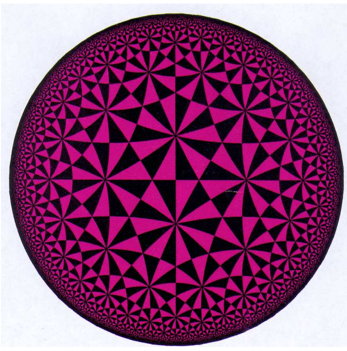
In the Colloquium organized two years ago in Maubeuge whose proceedings have just been published by Springer-Verlag [2], an outlook on this project was given, with a short description of some of the follies.

Very classical geometry will be illustrated by two of them. I shall only briefly describe one of these : it is named "the seventh temple", and refers to one of the most ancient artistic activity of men frequently bound with religion, the realization of friezes and tessellations, the construction of tombs and temples. It is connected to one of the most interesting physical question on the ways followed by Nature to fill up space.



First draft of the seventh temple made in 1991

The building will offer illustrations of the seven frieze groups (7 inner walls), of the seventeen wall-paper groups (17 outer walls), of tessellations of the hyperbolic plane (the cupola and the ground), of some braid groups (columns between the exterior walls), of various polyhedra (above the capitals of the columns).



The ($\pi/3$, $\pi/4$, $\pi/7$) Kleinian tessellation of the hyperbolic plane by triangles having the minimum area $\pi/42$

Artists, schoolboys as well, should be invited to contribute to the decoration of the interior and exterior walls. Owing to these illustrations, at different levels, through various kinds and means of explanation, people will be able to enter the geometry and the algebra issued from the study of these mathematical objects. The groups of isometries play the most important role in this study. At a first look they seem to be of static character. In order to show the importance of the notion of movement which is at the root of the concept of group, the various polyhedra around the cupola of this folly will be able to rotate along one of their symmetry axis - so that they will eventually be substituted by others.

Of course, one folly will illustrate classical the theory of curves of second order (conics), an other one differential geometry (quadrics, tangent spaces, minimal surfaces). An other folly, the famous Whitney umbrella, will illustrate algebraic geometry and important concepts as those of singularity and stratification.

Number theory and standard analysis will also be present, while a reproduction of Koenigsberg bridges will recall the birth of graph theory and the premises of topology.

Many such follies can be conceived. Various constraints have to be respected and have to be compatible. That is the case of the constraints of financial and physical feasibility which are consistent with some pedagogical choices. In order to allow our objects to have a good if not the best psychological impact, they must not be too much large, but not too much small either. Too large buildings are psychologically crushing, while too small objects are not enough impressive. Follies having the size of a common house seem to offer a good compromise. It is a size at a human scale. When one looks at such a folly with a convenient withdrawal, we can be impressed both by the whole size of the folly and by the presence of its visible significant details on which the eye may be attracted by various means : shapes of the details, their colour and brightness according to the material they are built in. Most follies are around 10 meters high, 60 square meters on the ground. They should disposed so that they be not visible one from each other. In that way, while leaving a folly to reach an other one, the activity of the mind can rest and be refreshed by the walk, being then prepared to be surprised by the discovering of a quite new folly. The aim of surprising the visitor is to induce a stronger imprinting of the folly in the mind, so that it will keep a better track of the mathematics involved into the folly.

The previous follies are tied with mathematics that have mostly been settled before the last century. Other follies have to be conceived in order to take into account more recent mathematics.

The following one is devoted to differential analysis and topological dynamics. The body of the folly is a representation of the topological 2-torus $\mathbb{T}^2$. Let us give a few reasons for this choice.

A topological space is defined with respect to a deformation called an homeomorphism. Given such a topological space, there are usually an infinite number of ways to represented it into a metrical space. While the 0-torus is always represented by a point, the 1-torus $\mathbf{T}^1$, which is also the 1-sphere $\mathbf{S}^1$, can be represented by the usual circle $\mathbf{C}^1$ of radius 1 in the Euclidean 2-space $\mathbb{R}^2$ ($\mathbf{S}^1 = \mathbf{T}^1$ is the topological space underlying the circle). Given a point O on this circle, and an orientation of the circle, any other point A on the circle is determined by the length $\theta_A$ of the oriented arc OA, which is called the angle between the two points. From a dynamical point of view, this angle has a deep physical meaning. Suppose that it takes one unit of time to run on the circle along OA at a constant speed. Then it will take $T_A = 2\pi/\theta_A$ units of time for the mobile body to be back at the same place. Inverses of angles have the meaning of periods, while angles have the meaning of frequencies.

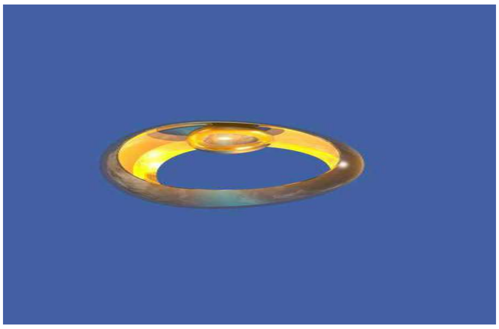
We must keep in mind that in Nature, objects are characterized by the presence of internal constituents with periodic modifications any time these constituents are stable. There might be only one constituent c characterized by the value ρ of a parameter evolving with the period T associated with the angle θ, in order words c is characterized by the couple (ρ, θ) which can be seen as an element of $\mathbf{P} \times \mathbf{C^1} = \mathbb{C}$ where **P** is the set of non negative real numbers, and $\mathbb{C}$ the set of Chuquet-Cardan or complex numbers – note that $\mathbf{C^1}$ can be identified with $\mathbf{1} \times \mathbf{C^1}$. More generally, there are several such constituents $c_1, c_2, \ldots, c_n$, each one being represented by an element of $\mathbb{C}$. The set of their frequencies is a significant set of parameters (belonging to $\mathbf{C^1} \times \mathbf{C^1} \times \ldots \times \mathbf{C^1}$) which can be frequently measured. In the case where n = 2, $\mathbf{C^1} \times \mathbf{C^1} = \mathbf{C^2}$ is a metrical representation of the underlying topological 2-torus $\mathbf{T^2} = \mathbf{S^1} \times \mathbf{S^1}$. I have thus chosen to focus on the torus $\mathbf{T^2}$ first for physical reasons.

A secondreason is the fact that any topological n-sphere $\mathbf{S^n}$ is a very special case of the n-torus $\mathbf{T^n}$. In some sense, a torus is an unfolding of a sphere. There is a geometrical way to show this fact, and a numerical way as well, for instance, when n = 2, through the mappings $\mathbf{T^2} \to \mathbf{C^2} \to \mathbf{\Sigma^2}$, $t \to (\theta_1, \theta_2) \to (\cos\theta_2 \cos\theta_1, \cos\theta_2 \sin\theta_1, \sin\theta_2)$ where $\mathbf{\Sigma^2}$ is the standard representation of the topological 2-sphere $\mathbf{S^2}$ in the usual Euclidean 3-space $\mathbb{R}^{\mathbf{3}}$.

A third reason is the fact that a 2-torus is the “ blowing up” of each point of a knot **k** into a 1-sphere $\mathbf{S^1}$, a knot being a closed curve in the previous 3-space whose underlying topological space is $\mathbf{S^1}$, so that $\mathbf{T^2} = \mathbf{k} \times \mathbf{S^1}$.

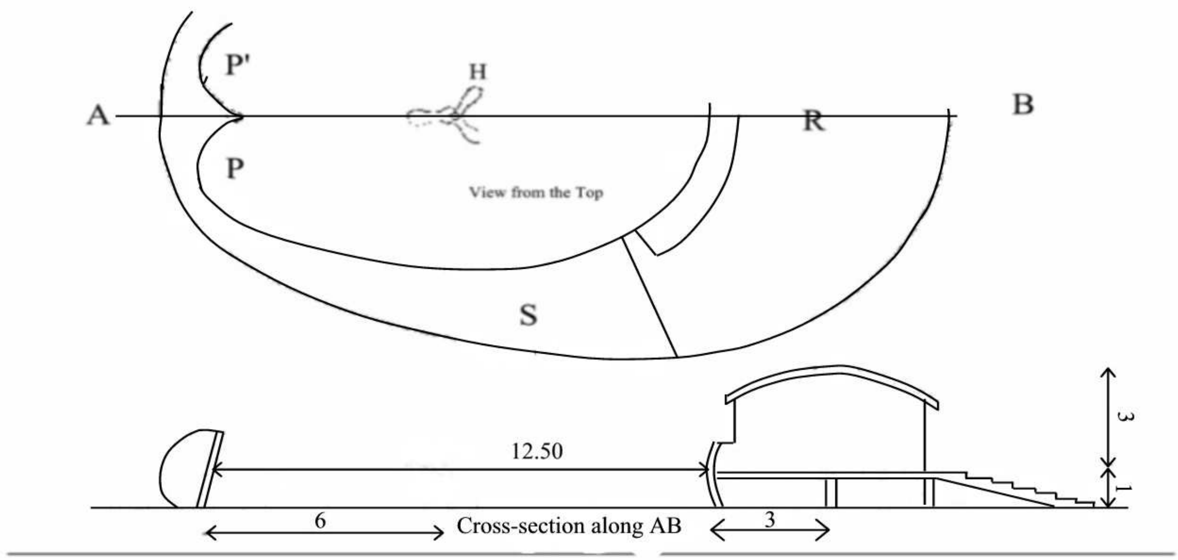
Leaving on a 2-sphere, we are familiar with the possible moves on this surface. Moves on a torus have also been explored, as for instance linear transformations of the torus into itself (called automorphisms). The effect of these automorphisms on small domains of the torus can be represented by nice plays of bands rolling around the torus. These bands, made with some noble metal, will participate to the decoration of the folly.

Usually the 2-torus is materialized into a doughnut, an inner tube or a ring. That will not be totally the case here since the torus will be used to illustrate some results in differential analysis through optics.



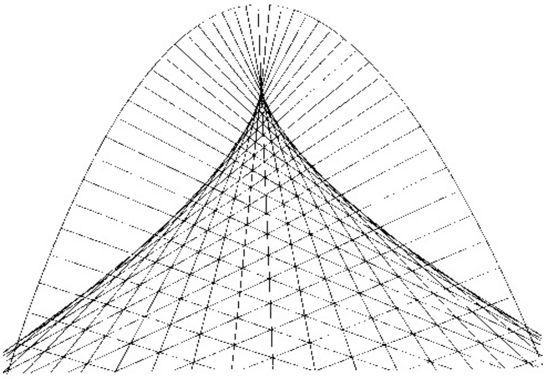
The “womb” : a sculpture made by John Robinson, a torus from the topological point of view

I have drawn all the follies in my head more than twelve years ago now, and I have given their names around the same time. The beautiful John’s Robinson sculpture he called “the womb” is also made very luminous on this image. The folly called “the luminous torus” is no too far from the womb. Since this folly has the number 10 on the list, I thought it was not necessary to hurry to prepare detailed plans. I shall thus only show here a first draft.




The folly is divided into two parts : the first one **R** is a large enough room where the public will stay, and from which he will be able to look at the effects produced by various lightings of the second part and inside this part.

As it should be well known, geometry comes from optics and optical effects, and can be viewed as an abstract physical theory of light. During these last 30 years, mathematicians, first H. Whitney and R. Thom, and physicists, in particular Berry [1], have developed theories explaining the appearance of caustics in optics. These caustics are surfaces along which luminous energy is concentrated. They occur by reflection or refraction of rays. Here is the simplest standard caustic obtained by reflection of rays on a parabolic mirror : at each point of the caustic, at least two rays are in contact. Two caustics of this type, called the “cusp”, will produced in the inner court of the folly, by reflexions on the mirrors **P** and **P’**. The colour of the ground which will serve as a screen will be adapted to the colour of the light beams striking the mirrors.



Less elementary caustics obtained by diffraction will be produced in the part **S** inside the folly.

A sculpture, made in glass with various indices, stays in the middle of the court. The light along the lines where the curvature of this sculpture changes again induces the presence of caustics.

It has the shape of an hand with three fingers and a hole at the level of the palm so that it represents again a topological torus. This sculpture may be seen also as a stylisation of a flower, the bright charm of the soft topology.

**Bibliography**

[1] M. BERRY , C. UPSTILL *Catastrophic Optics : Morphologies of Caustics and their Diffraction Patterns*, in Progress on Optics, vol. 18, pp 259-345, Elsevier- North Holland, 1988.

[2] C.P. BRUTER (Ed.) Mathematics and Art, Mathematical Visualization in Art and Education, Springer-Verlag, Heidelberg, 2002.